

Strict Triangular Form

Ex:

$$\begin{cases} 3x_1 + 2x_2 - x_3 = -2 \\ x_2 - 2x_3 = 3 \\ 2x_3 = 4 \end{cases}$$

$x_1 = \dots$
 $x_3 = \dots$

$$3x_1 + 2 \cdot 7 - 2 = -2 \Rightarrow x_1 = -14/3$$

$$x_2 - 2 \cdot 2 = 3 \Rightarrow x_2 = 7$$

substitute
 $x_3 = 2$

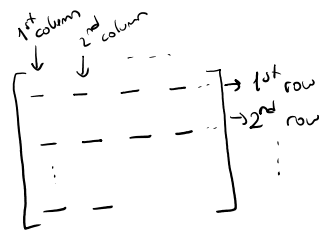
$\Rightarrow (-14/3, 7, 2)$ → the unique solution for the system.



Systems of Linear Eqns

Matrices

2D arrays of real numbers



m rows, n columns → m × n matrix

a_{ij} → the coeff of the j^{th} variable (x_j) in the i^{th} eqn

ex/ $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \end{bmatrix}$ 2 × 4 # of rows, # of columns

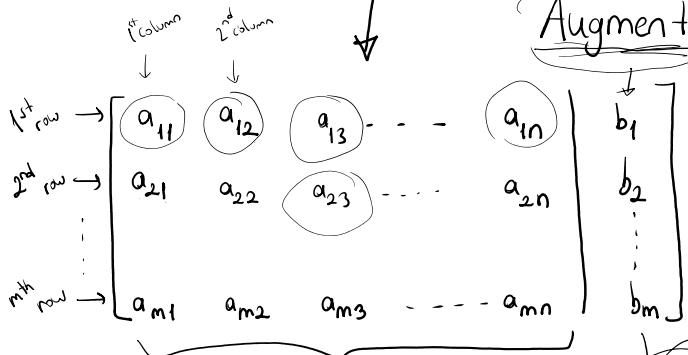
$$\begin{cases} 1^{\text{st}} \text{ eqn} \rightarrow a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ 2^{\text{nd}} \text{ eqn} \rightarrow a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ m^{\text{th}} \text{ eqn} \rightarrow a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

m eqns

n variables x_1, x_2, \dots, x_n

m × n system of linear eqns

Augmented Matrix of a System of LE



→ m rows, n + 1 column

m × (n + 1) matrix

coefficients

results column

$$(A = [a_{11} \dots a_{1n}])$$

coefficients

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

coefficient matrix

column

Ex/

$$\begin{aligned} 3x_1 + 2x_2 - 1x_3 &= -2 \\ 0x_1 + 1x_2 - 2x_3 &= 3 \\ 0x_1 + 0x_2 + 2x_3 &= 4 \end{aligned} \rightarrow$$

The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 2 & 4 \end{array} \right] \rightarrow \text{in strict triangular form}$$

Ex/

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \rightarrow \text{1st linear eq.} \\ -3x_1 - x_2 - x_3 &= 5 \rightarrow \text{2nd} \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}$$

- Allowed Operations to create "Equivalent" Systems**
- ! 1) changing the orders of any two eqns
 - ! 2) multiplying one eqn with a real number
 - 3) $c r_i + r_j \rightarrow r_j$! $c \in \mathbb{R}$

Ex/

the augmented matrix of the system

$$\begin{array}{l} r_1 \rightarrow \\ r_2 \rightarrow \\ r_3 \rightarrow \end{array} \left[\begin{array}{ccc|c} 3 & 2 & -1 & -2 \\ -3 & -1 & -1 & 5 \\ 3 & 2 & 1 & 2 \end{array} \right] \xrightarrow{r_2 \leftrightarrow r_3}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & -2 \\ 3 & 2 & 1 & 2 \\ -3 & -1 & -1 & 5 \end{array} \right] \rightarrow \text{an equivalent system to the original one.}$$

Ex/

$$r_1 \rightarrow \left[\begin{array}{ccc|c} 3 & 2 & -1 & -2 \\ -3 & -1 & -1 & 5 \\ 3 & 2 & 1 & 2 \end{array} \right] \xrightarrow{\frac{1}{3} \cdot r_1 \rightarrow r_1}$$

$$\left[\begin{array}{ccc|c} 1 & 2/3 & -1/3 & -2/3 \\ -3 & -1 & -1 & 5 \\ 3 & 2 & 1 & 2 \end{array} \right] \rightarrow \text{an equivalent system to the original one.}$$

$$\downarrow 2r_3 \rightarrow r_3$$

$$\left[\begin{array}{ccc|c} 1 & 2/3 & -1/3 & -2/3 \\ -3 & -1 & -1 & 5 \\ 6 & 4 & 2 & 4 \end{array} \right] \rightarrow \text{an equivalent system to the original one.}$$

$$\downarrow r_1 \leftrightarrow r_2$$

$$\left[\begin{array}{ccc|c} -3 & -1 & -1 & 5 \\ 1 & 2/3 & -1/3 & -2/3 \\ 6 & 4 & 2 & 4 \end{array} \right] \rightarrow //$$

EX/ $\rightarrow \begin{bmatrix} 3 & 2 & -1 & -2 \\ -3 & -1 & -1 & 5 \\ 3 & 2 & 1 & 2 \end{bmatrix} \xrightarrow{2r_1 + r_2 \rightarrow r_2} \begin{bmatrix} 3 & 2 & -1 & -2 \\ 3 & 3 & -3 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix} \rightarrow \text{an equiv. system}$

$\begin{matrix} 2r_1 & 6 & 4 & -2 & -4 \\ r_2 & -3 & -1 & -1 & 5 \end{matrix} \} \rightarrow \begin{bmatrix} 3 & 3 & -3 & 1 \end{bmatrix} \rightarrow r_2$

EX/ $\begin{bmatrix} 3 & 2 & -1 & -2 \\ -3 & -1 & -1 & 5 \\ 3 & 2 & 1 & 2 \end{bmatrix} \xrightarrow{1 \cdot r_1 + r_2 \rightarrow r_2} \begin{bmatrix} 3 & 2 & -1 & -2 \\ 0 & 1 & -2 & 3 \\ 3 & 2 & 1 & 2 \end{bmatrix}$

to make this zero
use multiples of the upper row.

to make this zero
use the nonzero upper rows.

$\begin{bmatrix} 3 & 2 & -1 & -2 \\ 0 & 1 & -2 & 3 \\ 3 & 2 & 1 & 2 \end{bmatrix} \xrightarrow{-1 \cdot r_1 + r_3 \rightarrow r_3} \begin{bmatrix} 3 & 2 & -1 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 2 & 4 \end{bmatrix} \rightarrow \text{an equiv.}$

EX/ 3) $c \cdot r_i + r_j \rightarrow r_j$
different r_j 's affected ✓
same r_i is used ✓

$\begin{bmatrix} 3 & 2 & -1 & -2 \\ -3 & -1 & -1 & 5 \\ 3 & 2 & 1 & 2 \end{bmatrix} \xrightarrow{\begin{matrix} 1r_1 + r_2 \rightarrow r_2 \\ -1r_1 + r_3 \rightarrow r_3 \end{matrix}} \begin{bmatrix} 3 & 2 & -1 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 2 & 4 \end{bmatrix}$